2011 FR

A graphing calculator is required for this problem

1. For $0 \le t \le 6$, a particle is moving along the x-axis. The particle's position, $x(t)$, is not explicitly given. The velocity of the particle is given by $v(t) = 2\sin\left(e^{t/4}\right) + 1$. The acceleration of the particle is given by $a(t) = \frac{1}{2}e^{t/4}\cos\left(e^{t/4}\right)$ and $x(0) = 2$.

   (a) Is the speed of the particle increasing or decreasing at time $t = 5.5$? Give a reason for your answer.

   (b) Find the average velocity of the particle for the time period $0 \le t \le 6$.

   (c) Find the total distance traveled by the particle from time $t = 0$ to $t = 6$.

   (d) For $0 \le t \le 6$, the particle changes direction exactly once. Find the position of the particle at that time.

a) $v(5.5) \approx -0.453$
$a(5.5) \approx -1.359$
} Since velocity and acceleration @ $t = 5.5$ are both negative (same sign), the speed is increasing.

b) Avg velocity $= \frac{1}{6-0}\int_0^6 \left[2\sin\left(e^{t/4}\right) + 1\right] dt$

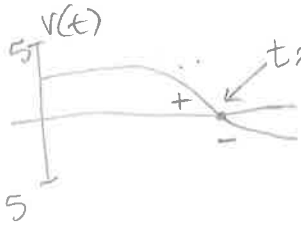
$\boxed{\approx 1.949}$

c) Total Distance $= \int_0^6 \left|2\sin\left(e^{t/4}\right) + 1\right| dt$ ← find absolute value

$\boxed{\approx 12.573}$

MATH ▶ NUM
abs(

d) when $v(t) = 0$ and changes sign.



$t \approx 5.196$

$x(\approx 5.196) = 2 + \int_0^{\approx 5.196} v(t)\, dt$

$\boxed{\approx 14.135}$

FR 2011

Calculator OK.

t (minutes)	0	2	5	9	10
$H(t)$ (degrees Celsius)	66	60	52	44	43

} unequal subintervals

2. As a pot of tea cools, the temperature of the tea is modeled by a differentiable function H for $0 \le t \le 10$, where time t is measured in minutes and temperature $H(t)$ is measured in degrees Celsius. Values of $H(t)$ at selected values of time t are shown in the table above.

(a) Use the data in the table to approximate the rate at which the temperature of the tea is changing at time $t = 3.5$. Show the computations that lead to your answer.

(b) Using correct units, explain the meaning of $\frac{1}{10}\int_0^{10} H(t)\,dt$ in the context of this problem. Use a trapezoidal sum with the four subintervals indicated by the table to estimate $\frac{1}{10}\int_0^{10} H(t)\,dt$.

Average Value

$A = \frac{h}{2}(b_1 + b_2)$

(c) Evaluate $\int_0^{10} H'(t)\,dt$. Using correct units, explain the meaning of the expression in the context of this problem.

(d) At time $t = 0$, biscuits with temperature 100°C were removed from an oven. The temperature of the biscuits at time t is modeled by a differentiable function B for which it is known that $B'(t) = -13.84e^{-0.173t}$. Using the given models, at time $t = 10$, how much cooler are the biscuits than the tea?

a) $H'(3.5) \approx \frac{H(5) - H(2)}{5 - 2}$

$\approx \frac{52 - 60}{3}$

$\approx -\frac{8}{3}$ °C per minute.

d) $(t, B(t)) \to (0, 100)$

$H(10) =$ Tea temp @ $t = 10 \to 43$°C

$B(10) = 100 + \int_0^{10} B'(t)\,dt$

$B(10) = 100 + \int_0^{10} (-13.84e^{-0.173t})\,dt$

≈ 34.183° C

$H(10) - B(10) \approx 8.817$

The biscuits are ≈ 8.817°C cooler than the tea at $t = 10$ minutes.

b) $\frac{1}{10}\int_0^{10} H(t)\,dt$ is the average temperature in °C of the tea over the time interval $0 \le t \le 10$ minutes.

$\frac{1}{10}\int_0^{10} H(t)\,dt \approx \frac{1}{10} \cdot \frac{1}{2}\left[2(H(0) + H(2)) + 3(H(2) + H(5)) + 4(H(5) + H(9)) + 1(H(9) + H(10))\right]$

$\approx \frac{1}{20}\left[2(126) + 3(112) + 4(96) + 87\right]$

$\boxed{\approx 52.95\text{ °C}}$

c) $\int_0^{10} H'(t)\,dt$ represents the accumulated change in temperature on the time interval $0 \le t \le 10$ minutes

$H(10) - H(0)$

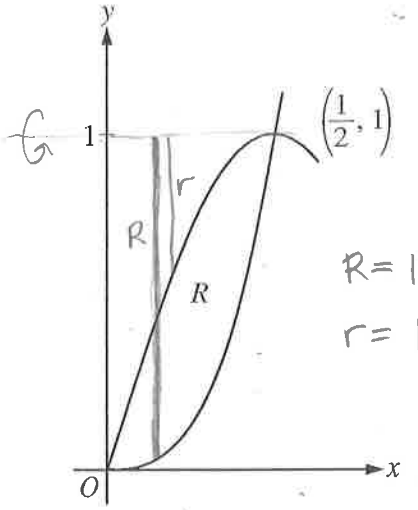
$43 - 66$

-23

The temperature decreased 23°C over the time interval $0 \le t \le 10$ minutes.

2011 FR

No calculator is allowed for these problems.



$R = 1 - 8x^3$
$r = 1 - \sin(\pi x)$

3. Let R be the region in the first quadrant enclosed by the graphs of $f(x) = 8x^3$ and $g(x) = \sin(\pi x)$, as shown in the figure above.

   (a) Write an equation for the line tangent to the graph of f at $x = \frac{1}{2}$.

   (b) Find the area of R.

   (c) Write, but do not evaluate, an integral expression for the volume of the solid generated when R is rotated about the horizontal line $y = 1$.

a) $f'(x) = 24x^2 \quad f\left(\frac{1}{2}\right) = 1 \quad$ tangent line: $y - 1 = 6\left(x - \frac{1}{2}\right)$
$f'\left(\frac{1}{2}\right) = 6$

b) $A = \int_0^{\frac{1}{2}} (\sin(\pi x) - 8x^3)\,dx$ $\quad u = \sin(\pi x)$
$du = -\pi\cos(\pi x)\,dx$

$= -\frac{1}{\pi}\cos(\pi x) - 2x^4 \Big]_0^{\frac{1}{2}}$

$= -\frac{1}{\pi}\cos\left(\frac{\pi}{2}\right) - \frac{1}{8} - \left(-\frac{1}{\pi}\cos(0)\right)$

$= -\frac{1}{\pi}(0) - \frac{1}{8} + \frac{1}{\pi}$

$\boxed{= \frac{1}{\pi} - \frac{1}{8}}$

c) $V = \pi\int_0^{1/2} \left[(1 - 8x^3)^2 - (1 - \sin(\pi x))^2\right] dx$

No Calculator

c) Point of Inflection where g' goes from increasing to decreasing at $x=0$.

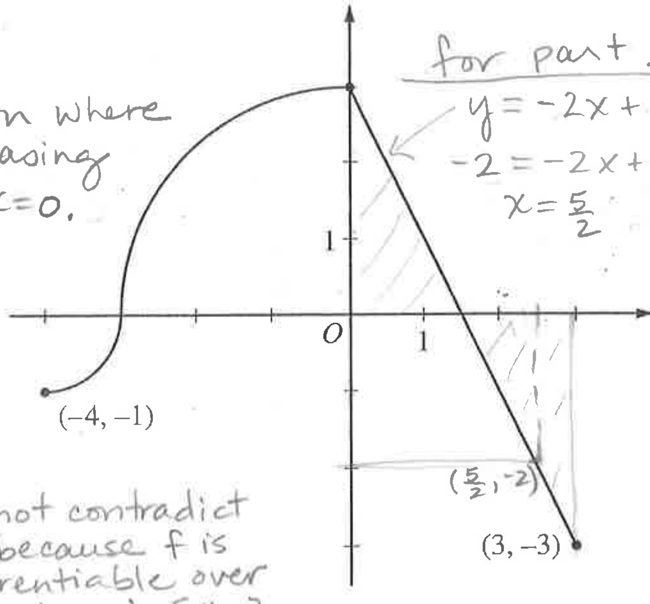
d) Avg. Rate of Change

$= \dfrac{f(3) - f(-4)}{3-(-4)}$

$= \dfrac{-3-(-1)}{7}$

$\boxed{= -\dfrac{2}{7}}$

It does not contradict the MVT because f is not differentiable over its entire domain $[-4,3]$ which is a criteria for MVT. ← f is not differentiable @ $x=-3$ or $x=0$



Graph of f

for part b:

$y = -2x+3$

$-2 = -2x+3$

$x = \frac{5}{2}$

4. The continuous function f is defined on the interval $-4 \le x \le 3$. The graph of f consists of two quarter circles and one line segment, as shown in the figure above. Let $g(x) = 2x + \int_0^x f(t)\,dt$.

(a) Find $g(-3)$. Find $g'(x)$ and evaluate $g'(-3)$.

(b) Determine the x-coordinate of the point at which g has an absolute maximum on the interval $-4 \le x \le 3$. Justify your answer.

(c) Find all values of x on the interval $-4 < x < 3$ for which the graph of g has a point of inflection. Give a reason for your answer.

(d) Find the average rate of change of f on the interval $-4 \le x \le 3$. There is no point c, $-4 < c < 3$, for which $f'(c)$ is equal to that average rate of change. Explain why this statement does not contradict the Mean Value Theorem.

a) $g(-3) = 2(-3) + \int_0^{-3} f(t)\,dt$

$= -6 - \int_{-3}^{0} f(t)\,dt$

$= -6 - \frac{1}{4}\pi(3)^2$

$\boxed{= -6 - \frac{9\pi}{4}}$

$g'(x) = 2 + \frac{d}{dx}\int_0^x f(t)\,dt$

$\boxed{g'(x) = 2 + f(x)}$ ←

$g'(-3) = 2 + f(-3)$

$= 2 + 0$

$\boxed{g'(-3) = 2}$

Since $g'(x)$ is the graph of f shifted up 2, it will go from + to − @ $x = \frac{5}{2}$

b) Absolute max @ endpt or where $g'(x) = 0$ and goes from + to −

$g(-4) = 2(-4) + \int_0^{-4} f(t)\,dt$

$= -8 - \int_{-4}^{0} f(t)\,dt$

$= -8 - \left[\frac{9\pi}{4} - \pi\right]$

$g(-4) = -8 - \frac{5\pi}{4}$

$g(3) = 6 + \int_0^3 f(t)\,dt$

$= 6 + 0$

$g(3) = 6$

$\boxed{\text{Absolute Max @ } x = \frac{5}{2}}$

$0 = 2 + f(x)$

$f(x) = -2$ @ $x = \frac{5}{2}$

$g(\frac{5}{2}) = 2(\frac{5}{2}) + \int_0^{5/2} f(t)\,dt$

$= 5 + \frac{1}{2}\cdot\frac{3}{2}(3) - \frac{1}{2}(1)(2)$

$= 4 + \frac{9}{4}$

$g(\frac{5}{2}) = \frac{25}{4} = 6.25$

5. At the beginning of 2010, a landfill contained 1400 tons of solid waste. The increasing function W models the total amount of solid waste stored at the landfill. Planners estimate that W will satisfy the differential equation $\frac{dW}{dt} = \frac{1}{25}(W - 300)$ for the next 20 years. W is measured in tons, and t is measured in years from the start of 2010.

(a) Use the line tangent to the graph of W at $t = 0$ to approximate the amount of solid waste that the landfill contains at the end of the first 3 months of 2010 (time $t = \frac{1}{4}$).

(b) Find $\frac{d^2W}{dt^2}$ in terms of W. Use $\frac{d^2W}{dt^2}$ to determine whether your answer in part (a) is an underestimate or an overestimate of the amount of solid waste that the landfill contains at time $t = \frac{1}{4}$.

(c) Find the particular solution $W = W(t)$ to the differential equation $\frac{dW}{dt} = \frac{1}{25}(W - 300)$ with initial condition $W(0) = 1400$.

a) $t=0$ in 2010 $\rightarrow$ $W = 1400$ tons

slope @ $t=0$ $\frac{dW}{dt} = \frac{1}{25}(1400-300)$

$= \frac{1100}{25}$

$= 44$

tangent line:

$W - 1400 = 44(t-0)$

$W = 44t + 1400$

@ $t = \frac{1}{4}$: $W \approx 44\left(\frac{1}{4}\right) + 1400$

$\boxed{W \approx 1411 \text{ tons}}$

b) $\frac{d^2W}{dt^2} = \frac{1}{25} \cdot \frac{dW}{dt}$

$= \frac{1}{25} \cdot \frac{1}{25}(W-300)$

@ $t = \frac{1}{4}$ $\rightarrow$ $\frac{d^2W}{dt^2} = \frac{1}{625}(1411-300)$

A positive 2nd derivative so W(t) is concave up, slopes are increasing at $t = \frac{1}{4}$

The tangent line at t=0 will give us an underestimation.

c) $\int \left(\frac{1}{W-300}\right) dW = \int \frac{1}{25} dt$

$\ln(W-300) = \frac{1}{25}t + C$

$\ln(1400-300) = \frac{1}{25}(0) + C$

$C = \ln(1100)$

$\ln(W-300) = \frac{1}{25}t + \ln(1100)$

$\ln\left(\frac{W-300}{1100}\right) = \frac{t}{25}$

$e^{(t/25)} = \frac{W-300}{1100}$

$\boxed{W(t) = 1100e^{(t/25)} + 300 \text{ for } 0 \leq t \leq 20}$

6. Let f be a function defined by $f(x) = \begin{cases} 1 - 2\sin x & \text{for } x \le 0 \\ e^{-4x} & \text{for } x > 0. \end{cases}$

(a) Show that f is continuous at $x = 0$.

(b) For $x \neq 0$, express $f'(x)$ as a piecewise-defined function. Find the value of x for which $f'(x) = -3$.

(c) Find the average value of f on the interval $[-1, 1]$.

a) $1 - 2\sin(0) \stackrel{?}{=} e^{-4(0)}$

$1 - 2(0) \stackrel{?}{=} e^0$

$1 = 1$ ✓

The 2 pieces connect @ (0,1)

$\lim_{x \to 0^+} [e^{-4x}] = 1$ & $\lim_{x \to 0^-} [1 - 2\sin x] = 1$

$\therefore f$ is continuous @ $x = 0$

b) $f'(x) = \begin{cases} -2\cos x, & x < 0 \\ -4e^{-4x}, & x > 0 \end{cases}$

$-3 = -4e^{-4x}$

$\frac{3}{4} = e^{-4x}$

$\frac{-4x}{-4} = \frac{\ln\left(\frac{3}{4}\right)}{-4}$

$\boxed{x = -\frac{1}{4}\ln\left(\frac{3}{4}\right)}$

$-3 = -2\cos x$

$\frac{3}{2} \neq \cos x$

c) Avg. value $= \frac{1}{1-(-1)} \int_{-1}^{1} f(x)\,dx$

$= \frac{1}{2}\left[\int_{-1}^{0} (1 - 2\sin x)\,dx + \int_{0}^{1} e^{-4x}\,dx\right]$

$= \frac{1}{2}\left[x + 2\cos x\right]_{-1}^{0} + \frac{1}{2}\left(-\frac{1}{4}\right)\left[e^{-4x}\right]_{0}^{1}$

$= \frac{1}{2}\left[0 + 2\cos(0) - (-1 + 2\cos(-1))\right] - \frac{1}{8}\left(e^{-4} - e^0\right)$

$= \frac{1}{2}\left(2 + 1 - 2\cos(-1)\right) - \frac{1}{8e^4} + \frac{1}{8}$

$= \frac{3}{2} - \cos(-1) - \frac{1}{8e^4} + \frac{1}{8}$

$\boxed{= \frac{13}{8} - \cos(-1) - \frac{1}{8e^4}}$