2012 Key

A graphing calculator is required for these problems.

unequal subintervals

t (minutes)	0	4	9	15	20
$W(t)$ (degrees Fahrenheit)	55.0	57.1	61.8	67.9	71.0

1. The temperature of water in a tub at time t is modeled by a strictly increasing, twice-differentiable function W, where $W(t)$ is measured in degrees Fahrenheit and t is measured in minutes. At time $t = 0$, the temperature of the water is 55°F. The water is heated for 30 minutes, beginning at time $t = 0$. Values of $W(t)$ at selected times t for the first 20 minutes are given in the table above.

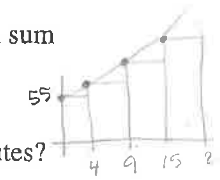
$\to W'$ is $+$

(a) Use the data in the table to estimate $W'(12)$. Show the computations that lead to your answer. Using correct units, interpret the meaning of your answer in the context of this problem. Avg. Rate of Change p. 105 MVT p. 164.

(b) Use the data in the table to evaluate $\int_0^{20} W'(t)\,dt$. Using correct units, interpret the meaning of $\int_0^{20} W'(t)\,dt$ in the context of this problem.

(c) For $0 \le t \le 20$, the average temperature of the water in the tub is $\frac{1}{20}\int_0^{20} W(t)\,dt$. Use a left Riemann sum with the four subintervals indicated by the data in the table to approximate $\frac{1}{20}\int_0^{20} W(t)\,dt$. Does this approximation overestimate or underestimate the average temperature of the water over these 20 minutes? Explain your reasoning.

(Avg. Value p. 264)



(d) For $20 \le t \le 25$, the function W that models the water temperature has first derivative given by $W'(t) = 0.4\sqrt{t}\cos(0.06t)$. Based on the model, what is the temperature of the water at time $t = 25$?

a) Estimate $W'(12) \approx \dfrac{W(15) - W(9)}{15 - 9} = \dfrac{67.9 - 61.8}{6} \approx 1.017\,°F/\text{min}$

At time $t = 12$ min, the temp. of the water is increasing at a rate of $\approx 1.017°F/\text{min}$.

b) $\displaystyle\int_0^{20} W'(t)\,dt \approx W(20) - W(0) = 71.0 - 55.0 = 16°F$

The net change in the temperature of the water is 16°F warmer over the time interval $0 < t < 20$ min.

c) $\displaystyle\frac{1}{20}\int_0^{20} W(t)\,dt \approx \frac{1}{20}\left[4(W(0)) + 5(W(4)) + 6(W(9)) + 5(W(15))\right]$

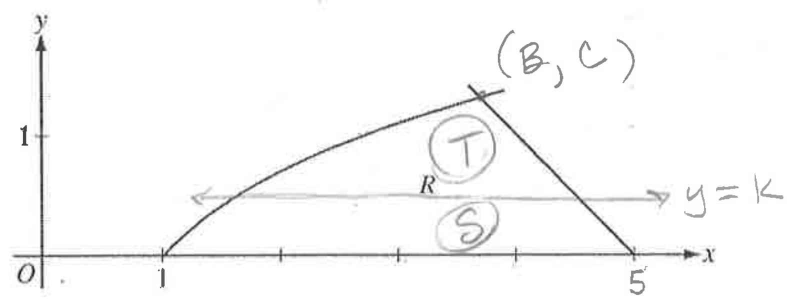
$= \frac{1}{20}\left[4(55) + 5(57.1) + 6(61.8) + 5(67.9)\right]$

$= \frac{1}{20}(1215.8) = 60.79°F$ is an underestimation of the avg. temp over $0 \le t \le 20$ min. Left endpts of an increasing function make all the rectangles under the curve.

d) $\displaystyle W(25) = 71 + \int_{20}^{25} W'(t)\,dt$

$\approx 73.043°F$

Calculator required



2. Let R be the region in the first quadrant bounded by the x-axis and the graphs of $y = \ln x$ and $y = 5 - x$, as shown in the figure above.

   (a) Find the area of R.

   (b) Region R is the base of a solid. For the solid, each cross section perpendicular to the x-axis is a square. Write, but do not evaluate, an expression involving one or more integrals that gives the volume of the solid.

   (c) The horizontal line $y = k$ divides R into two regions of equal area. Write, but do not solve, an equation involving one or more integrals whose solution gives the value of k.

a) Let B = x coordinate of intersection point: $\ln x = 5 - x$

$B \approx 3.693$

$$\text{Area of } R = \int_1^B \ln x \, dx + \int_B^5 (5-x)\,dx \approx \boxed{2.986} \leftarrow \left(\text{OR: } \int_0^C (5 - y - e^y)\,dy\right)$$

Let $R \approx 2.986$

b) $$\text{Volume} = \int_1^B (\ln x)^2 dx + \int_B^5 (5-x)^2 dx$$

c) $y = \ln x \rightarrow x = e^y$

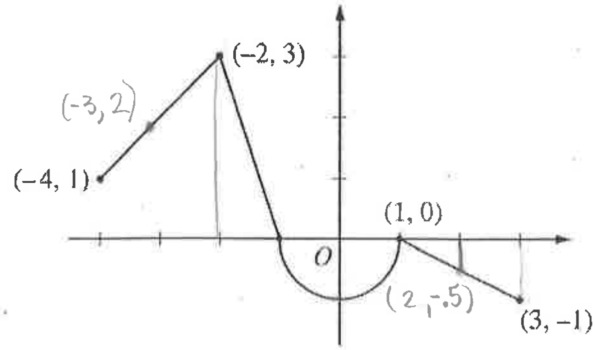
$y = 5 - x \rightarrow x = 5 - y$

$$\text{Area}_S = \frac{1}{2}\text{Area}_R$$

$$\int_0^k [(5-y) - e^y]\,dy = \frac{1}{2}(R)$$

or $\text{Area}_S = \text{Area}_T$ ...

No calculator is allowed for these problems.



Graph of $f = g'(x)$

3. Let f be the continuous function defined on $[-4, 3]$ whose graph, consisting of three line segments and a semicircle centered at the origin, is given above. Let g be the function given by $g(x) = \int_1^x f(t)\,dt$.

$g'(x) = \frac{d}{dx}\int_1^x f(t)dt = f(x)$

(a) Find the values of $g(2)$ and $g(-2)$.

(b) For each of $g'(-3)$ and $g''(-3)$, find the value or state that it does not exist.

(c) Find the x-coordinate of each point at which the graph of g has a horizontal tangent line. For each of these points, determine whether g has a relative minimum, relative maximum, or neither a minimum nor a maximum at the point. Justify your answers.

(d) For $-4 < x < 3$, find all values of x for which the graph of g has a point of inflection. Explain your reasoning.

just look @ area under the curve.

a) $g(2) = \int_1^2 f(t)dt = -\frac{1}{2}(1)\left(\frac{1}{2}\right) = \boxed{-\frac{1}{4}}$

$g(-2) = \int_1^{-2} f(t)dt = -\int_{-2}^{1} f(t)dt$

$= -\left[\int_{-2}^{-1} f(t)dt + \int_{-1}^{1} f(t)dt\right]$

$= -\left[\frac{1}{2}(1)(3) + \frac{1}{2}\pi(1)^2\right] = \boxed{\frac{\pi}{2} - \frac{3}{2}}$

b) $\boxed{g'(-3) = 2}$ ← value on graph @ x = -3

$\boxed{g''(-3) = 1}$ ← slope of graph @ x = -3

c) where $g' = 0$ and changes sign $\left\{ g'_{+} \rightarrow g'_{-} \right\}$ max @ $x = -1$

$g' = 0$ at $x = 1$, but g' doesn't change sign there so not an extrema.

d) POI where $g''(x)$ changes sign @ $x = -2$, $x = 0$, $x = 1$

No Calculator

4. The function f is defined by $f(x) = \sqrt{25 - x^2}$ for $-5 \le x \le 5$.

(a) Find $f'(x)$.

(b) Write an equation for the line tangent to the graph of f at $x = -3$.

(c) Let g be the function defined by $g(x) = \begin{cases} f(x) & \text{for } -5 \le x \le -3 \\ x + 7 & \text{for } -3 < x \le 5. \end{cases}$

Is g continuous at $x = -3$? Use the $\underbrace{\text{definition of continuity}}_{\text{p. 71}}$ to explain your answer.

(d) Find the value of $\int_0^5 x\sqrt{25 - x^2}\, dx$.

a) $f'(x) = \frac{1}{2}(25-x^2)^{-1/2}(-2x) = \boxed{-\frac{x}{\sqrt{25-x^2}},\ -5 < x < 5}$

b) $f'(-3) = -\frac{(-3)}{\sqrt{25-9}} = \frac{3}{4}$ $\quad \boxed{y - 4 = \frac{3}{4}(x+3)}$ $\quad y = \frac{3}{4}x + \frac{25}{4}$

$f(-3) = \sqrt{25-9} = 4$

c) $g(-3) = f(-3) = 4$ and $-3+7 = 4$ } yes g is continuous @ $x=-3$ b/c $\lim_{x\to -3^+} g(x) = \lim_{x \to -3^-} g(x) = g(-3)$

d) $\int_0^5 x\sqrt{25-x^2}\,dx$ $\quad u = 25 - x^2$ $\quad x = 0 \to u = 25$

$du = -2x\,dx$ $\quad x = 5 \to u = 0$

$-\frac{1}{2}\int_0^5 -2x\sqrt{25-x^2}\,dx$

$= -\frac{1}{2}\int_{25}^0 \sqrt{u}\,du$

$= \frac{1}{2}\int_0^{25} u^{1/2}\,du$

$= \frac{1}{2}\left[\frac{2u^{3/2}}{3}\right]_0^{25} = \frac{1}{3}\left[25^{3/2} - 0^{3/2}\right] = \boxed{\frac{125}{3}}$

✓ Calculator

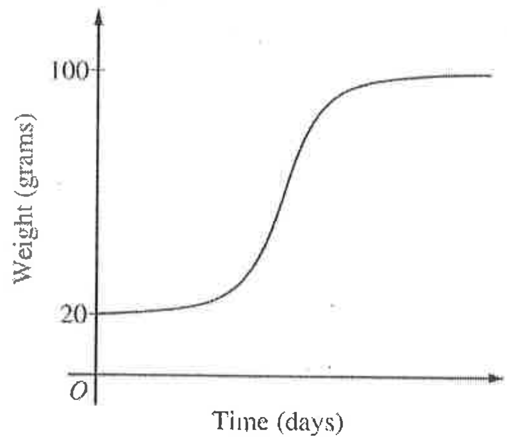
5. The rate at which a baby bird gains weight is proportional to the difference between its adult weight and its current weight. At time $t = 0$, when the bird is first weighed, its weight is 20 grams. If $B(t)$ is the weight of the bird, in grams, at time t days after it is first weighed, then

$$\frac{dB}{dt} = \frac{1}{5}(100 - B).$$

Let $y = B(t)$ be the solution to the differential equation above with initial condition $B(0) = 20$. (t, B) $(0, 20)$

(a) Is the bird gaining weight faster when it weighs 40 grams or when it weighs 70 grams? Explain your reasoning. compare rates, $\frac{dB}{dt}$, @ 40 & 70.

(b) Find $\frac{d^2B}{dt^2}$ in terms of B. Use $\frac{d^2B}{dt^2}$ to explain why the graph of B cannot resemble the following graph.



(c) Use separation of variables to find $y = B(t)$, the particular solution to the differential equation with initial condition $B(0) = 20$.

a) $B'(40) = \frac{1}{5}(100-40)$ $= 12$ grams/day

$B'(70) = \frac{1}{5}(100-70)$ $= 6$ grams/day

At 12 grams/day, the bird is gaining weight faster when it weighs 40 grams than 6 grams/day when it weighs 70 gr.

b) $\frac{d^2B}{dt^2} = -\frac{1}{5}\frac{dB}{dt}$ $\quad 20 \le B < 100$

$= -\frac{1}{5}\cdot\frac{1}{5}(100-B)$, Since the 2nd derivative is negative, the graph of B should be concave down over the entire domain. The graph shown is not.

c) $\int \frac{dB}{100-B} = \int \frac{1}{5}dt$ $\quad u = 100-B$, $du = -dB$

$-\int -\frac{1}{100-B}\,dB = \frac{1}{5}\int dt$

* 100−B>0 don't need abs. value

$-\ln^{*}(100-B) = \frac{t}{5} + C$

$(0,20) \rightarrow$ $-\ln(100-20) = \frac{0}{5} + C$

$C = -\ln 80$

$-\ln(100-B) = \frac{t}{5} - \ln 80$

$\rightarrow \ln 80 - \ln(100-B) = \frac{t}{5}$

$\ln\left(\frac{80}{100-B}\right) = \frac{t}{5}$

$e^{t/5} = \frac{80}{100-B}$

$100 - B = \frac{80}{e^{t/5}}$

$100 - \frac{80}{e^{t/5}} = B$

$\boxed{B(t) = 100 - \frac{80}{e^{t/5}}, \quad t \ge 0}$

No Calculator

6. For $0 \le t \le 12$, a particle moves along the x-axis. The velocity of the particle at time t is given by $v(t) = \cos\left(\frac{\pi}{6}t\right)$. The particle is at position $x = -2$ at time $t = 0$.

(a) For $0 \le t \le 12$, when is the particle moving to the left?

(b) Write, but do not evaluate, an integral expression that gives the total distance traveled by the particle from time $t = 0$ to time $t = 6$.

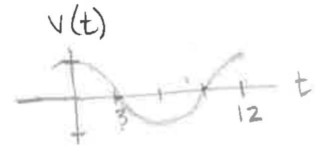
(c) Find the acceleration of the particle at time t. Is the speed of the particle increasing, decreasing, or neither at time $t = 4$? Explain your reasoning.

(d) Find the position of the particle at time $t = 4$.

a) Left when $v(t)$ is negative.

$3 < t < 9$

$\text{per} = \frac{2\pi}{\frac{\pi}{6}} = 12$

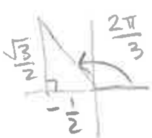


b) Total distance $= \int_0^6 \left|\cos\left(\frac{\pi}{6}t\right)\right| dt$

c) $a(t) = v'(t) = -\frac{\pi}{6}\sin\left(\frac{\pi}{6}t\right)$

$a(4) = -\frac{\pi}{6}\sin\left(\frac{\pi}{6}\cdot 4\right)$

$= -\frac{\pi}{6}\sin\left(\frac{2\pi}{3}\right)$



$v(4) = \cos\frac{\pi}{6}(4)$

$= -\frac{1}{2}$

$= -\frac{\pi}{6}\frac{\sqrt{3}}{2}$

Since both velocity and acceleration are negative at $t=4$, the speed of the particle is increasing.

d) $s(4) = -2 + \int_0^4 \cos\left(\frac{\pi}{6}t\right)dt$

$= -2 + \frac{6}{\pi}\int_0^4 \frac{\pi}{6}\cos\left(\frac{\pi}{6}t\right) dt$

$= -2 + \frac{6}{\pi}\left[\sin\frac{\pi}{6}t\right]_0^4$

$= -2 + \frac{6}{\pi}\left(\sin\frac{2\pi}{3} - \sin 0\right)$

$= -2 + \frac{6}{\pi}\left(\frac{\sqrt{3}}{2} - 0\right)$

$= \boxed{-2 + \frac{3\sqrt{3}}{\pi}}$